FUNCTIONS OF SEVERAL VARIABLES

(M.Sc., Paper-VI)

(Real Analysis-II)

Topic: - Topology of R^n, Limit & continuity, Derivative of Scalar filed w.r.t. a vector, Mean Value Theorem, Directional Derivative, The total derivative (Scalar field), Sufficient condition of differentiability, Derivative of a vector field,

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Functions of Two Variables

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$:→

A linear transformations

$$T : V = \mathbb{R}^n(\mathbb{R}) \longrightarrow W = \mathbb{R}^m(\mathbb{R}).$$

from a vector space V into another vector space W. (Where V and W are finite - dimensional vector space)

→ If $n = m = 1$ i.e; $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called a real - valued function of a real variable.

→ When $n = 1$, and $m > 1$ i,e; $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ is a vector valued function of a real variable.

→ When $n > 1$ and $m \geq 1$.

(i) If $m = 1$, the function is called a real-valued function of a vector variable or a scàlar field. (i,e; $f : \mathbb{R}^n \longrightarrow \mathbb{R}$)

(ii) If $m > 1$ it is called a vector-valued function of a vector variable or simply a vector field. (i,e; $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$).

Definition: → If $x, y \in \mathbb{R}^n$, then inner Product is defined as

If $x = (x_1, \dots, x_n)$ & $y = (y_1, \dots y_n) \in \mathbb{R}^n$

$$\langle x, y\rangle = x \cdot y = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + \dots + x_n y_n.$$

And corresponding norm is denoted by $\|x\|$ and defined as

$$\|x\| = \sqrt{\langle x, x\rangle} = (x \cdot x)^{\frac{1}{2}} = \sqrt{x_1^2 + \dots + x_n^2}.$$

Open ball and Open set: →

Let $a = (a_1, \dots, a_n)$ be a given point in $\mathbb{R}^n$ and $r > 0$ be a given positive real number.

The set of all points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ s.t.

$$\|x - a\| < r$$

is called an open n-ball of radius r and center a.

We denote this set by $B(a)$ or $B(a, r)$.

Example: → (i) In $\mathbb{R}$ this is simply an open interval $(a-r, a+r)$.

i.e; $B(a; r) = (a-r, a+r)$.

(ii) In $\mathbb{R}^2$ it is a circular disk

i.e; $B(a,r) = \{ x=(x_1,y_1) \in \mathbb{R}^2 : \|x-a\| < r \}$

Where $a = (a_1, a_2) \in \mathbb{R}^2$.

Then $\|x-a\| = \|(x_1-a_1, x_2-a_2)\|$

$= \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2}$

i.e; $B(a,r) = \{ x \in \mathbb{R}^2 : (x_1-a_1)^2 + (x_2-a_2)^2 < r^2 \}$

(iii) In $\mathbb{R}^3$ it is a spherical solid with center $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ of radius $r > 0$.

Definition (interior Point): → Let S be a subset of $\mathbb{R}^n$, and assume that $a \in S$. Then a is called an interior point of S if there is an open n-ball with center a, such that $B(a,r) \subseteq S$.

Remark: → ① The set of all interior points of S is called interior of S, and denoted by $\text{int}\, S$.

② An open set containing a point 'a' is sometimes called neighbourhood of 'a'

Definition (OPEN SET) :→ A set S in $\mathbb{R}^n$ is called open if all its points are interior points.

OR

$S \subseteq \mathbb{R}^n$ is open if and only if $S = \text{int} S$.

Definition (Exterior and Boundary) :→ A point $a \in \mathbb{R}^n$ is said to be exterior point of a set S in $\mathbb{R}^n$ if $\exists$ an open ~~ball~~ n-ball $B(a)$ containing no points of S.

The set of all points in $\mathbb{R}^n$ exterior to S is called the exterior of S and denoted by ext S.

A point which is neither exterior to S nor an interior point of S is called a boundary point of S.

The set of all ~~points of~~ boundary points of S is called the boundary of S and is denoted by ∂S.

Functions of two variables

Limits and continuity :→

Let S is a subset of $\mathbb{R}^n$. consider a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and if $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \dots, b_m) \in \mathbb{R}^m$

Then
$$\lim_{x \to a} f(x) = b \quad (\text{or } f(x) \to b \text{ as } x \to a) \quad -①$$

to mean that
$$\lim_{\|x-a\| \to 0} \|f(x) - b\| = 0. \quad —②$$

(It is not required that f be defined at the point a)

If we write $h = x - a$. Equation ② becomes
$$\lim_{\|h\| \to 0} \|f(x+h) - b\| = 0$$

→ A function f is said to be continuous at 'a' if f is defined at a and if
$$\lim_{x \to a} f(x) = f(b)$$

We say f is continuous on a set S if f is continuous at each point of S.

Functions of two variables

Derivative of a scalar field w.r.to a vector.

Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field.

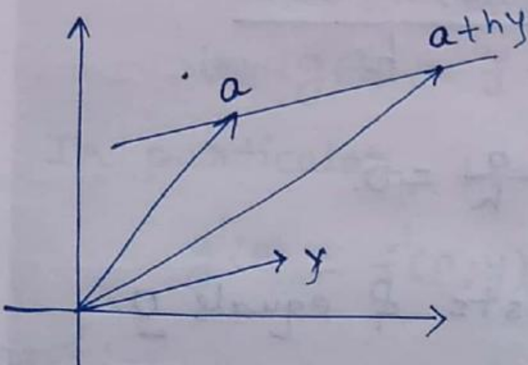
Let $\underline{a} = (a_1, \dots, a_n)$ be an interior point of S. Suppose we move from '$\underline{a}$' towords another point $\underline{a+y}$ along the line segment joining $\underline{a}$ and $\underline{a+y}$.

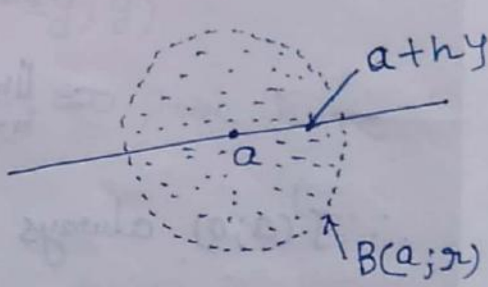
Each point of this segment is of the form $a+hy$, where h is a real number.

The distance from $\underline{a}$ to $\underline{a+hy}$ is $\|hy\| = |h| \|y\|$.

$\because$ '$\underline{a}$' is a interior point of S, there is an n-ball $B(a;r)$ laying intirely in S.

If h is chosen so that $\|hy\| = |h| \|y\| < r$, the ~~seg~~ segment from $\underline{a}$ to $\underline{a+hy}$ will lie in S.





We keep $h = (h_1, \dots h_n) \neq 0$ but small enough to guarantee that $\underline{a+hy \in S}$.

Definition of the derivative of a scalar field w.r.to a vector :→

Given a scalar field $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

Let a be an interior point of S and y be an arbitrary point in $\mathbb{R}^n$.

The derivative of f at a w.r.to y is denoted by the symbol $f'(a;y)$ and is defined by

$$f'(a,y) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h} \quad (\text{if exists})$$

Example ①→ If $y = 0$,

Then

$$f'(a,0) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$\therefore f'(a;0)$ always exists & equals $\underline{0}$.

Example ②:- Derivative of a linear transformation.

If $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is linear, then

$f(a+hy) = f(a) + h f(y)$, Then

$$f'(a,y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a) + hf(y) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{hf(y)}{h}$$

$$= \lim_{h\to 0} f(y) = f(y)$$

$\therefore f'(a,y) = f(y)$ for every $a \in S$ & every y in $\mathbb{R}^n$.

<u>THEOREM</u> :→ Let $g(t) = f(a+ty)$. If one of the ~~following~~ derivatives $g'(t)$ or $f'(a+ty;y)$ exists then the other exists and they are <u>equal</u>.

i.e; $g'(t) = f'(a+ty;y)$

In particular, when $t=0$, we have

$$g'(0) = f'(a;y)$$

<u>Proof</u>:→

$$\because \frac{g(t+h) - g(t)}{h} = \frac{f(a+ty+hy) - f(a+th)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{f'(a+ty+hy) - f(a+th)}{h}$$

$\therefore g'(t) = f'(a+ty;y)$ proved. — ②

putting $t=0$ in ②, we have

$$g'(0) = f'(a;y)$$

proved.

Example ③:→ Compute $f'(a;y)$ if $f(x) = \|x\|^2, \forall x \in \mathbb{R}^n$.

Solution:→

Let $g(t) = f(a+ty) = (a+ty)\cdot(a+ty)$
$$= a\cdot a + 2t\, a\cdot y + t^2 y\cdot y$$

$$\therefore g'(t) = 2a\cdot y + 2t\, y\cdot y$$

$$\Rightarrow g'(0) = 2a\cdot y$$

Ans.

THEOREM:→ (MEAN – VALUE THEOREM FOR DERIVATIVES OF SCALAR FIELDS)

Assume the derivative $f'(a+ty;y)$ exists for each t in the interval $0 \leq t \leq 1$. Then for some real θ in the open interval $0<\theta<1$. We have

$$f(a+y) - f(a) = f'(z;y) \quad \text{Where } z = a+\theta y.$$

Proof:→ Let $g(t) = f(a+ty)$.

Applying the Mean value theorem to g on the interval $[0,1]$ we have

$$g(1) - g(0) = g'(\theta)\,; \quad \text{Where } 0<\theta<1.$$

$$\because g(1) - g(0) = f(a+y) - f(a)$$

and $g'(\theta) = f'(a+\theta y\,;y)$.

$$\therefore f(a+y) - f(a) = f'(z;y) \quad \text{Where } z = a+\theta y$$

proved.

Functions of two variable.

Directional derivative and Partial derivatives:→

When y is a unit vector, that is $\|y\| = 1$, the distance between a and $a+hy$ is $|h|$, then

$$f'(a;y) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h} = (\nabla f)(a) \cdot y$$

Then $f'(a;y)$ is called the directional derivative of f at a, in the direction of y.

Remark:→ $\frac{f(a+hy) - f(a)}{h}$ represents the average rate of change of per unit distance along the line segment joining a to $a+hy$.

(The derivative $f'(a;y)$ is called directional derivative)

Remark:→ If $y = e_k$ (the kth unit coordinate vector) the directional derivative $f'(a;e_k)$ is called the partial derivative w.r.to. e_k. and is also denoted by the symbol $D_k f(a)$. Thus

$$D_k f(a) = f'(a;e_k)$$

Remark : → (i) If $f : S \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a ~~scalar~~ scalar fied.

Let $y = (y_1, y_2)$ be any unit vector.

Then $\|y\| = \sqrt{y_1^2 + y_2^2} = 1$. Let $a = (a_1, a_2) \in S$

Then $$f'(a; y) = \lim_{h \to 0} \frac{f(a + hy) - f(a)}{h}$$

$$= \lim_{(h_1, h_2) \to (0,0)} \frac{f(a_1 + h y_1, a_2 + h y_2) - f(a_1, a_2)}{h}$$

= Directional derivative of f at a in the direction of y.

(ii) The unit co-ordinate vectors are $e_1 = (1, 0)$ and $e_2 = (0, 1)$, where $\|e_1\| = \|e_2\| = 1$.

$$\therefore D_1 f(a) = f'(a; e_1) = \lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h}$$

and

$$D_2 f(a) = f'(a; e_2) = \lim_{h \to 0} \frac{f(a_1, a_2 + h) - f(a_1, a_2)}{h}$$

Where $D_1 f(a)$ and $D_2 f(a)$ are called the partial derivatives of f at a.

The Gradient :→ If $f(x,y,z)$ is a real-valued function of three variables, its gradient, which is denoted by ∇f or grad f, is defined by

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

For $f(x,y)$ the gradient is

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Remark :→ If y is a unit vector $(\|y\| = 1)$, then y specifies a direction in a plane or space, and we ~~calle~~ call $\nabla f(x)\cdot y$ the directional derivative of f at the point x in the direction of y.

Example :→ A scalar field f is ~~denoted by~~ defined on $\mathbb{R}^n$ by the equation $f(x) = a\cdot x$ where a is a constant vector. Compute $f'(x;y)$ for arbitrary x & y.

Solution :→

$$\because f'(x;y) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a\cdot(x+hy) - a\cdot x}{h}$$

$$= \lim_{h\to 0} \frac{a\cdot x + h(a\cdot y) - a\cdot x}{h}$$

$$= \lim_{h\to 0} \frac{h(a\cdot y)}{h}$$

$$= \lim_{h\to 0} (a\cdot y) = a\cdot y$$

$$\therefore f'(x;y) = a\cdot y$$

Answer.

Example ②: → A scalar field f defined on $\mathbb{R}^n$ by $f(x) = \|x\|^4$. Compute $f'(x;y)$ for arbitrary x and y.

Solution: →

$$\because f'(x;y) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\|(x+hy)\|^4 - \|x\|^4}{h}$$

Let $g(t) = f(x+ty) = \{(x+ty)\cdot(x+ty)\}^2$

$$= \{x\cdot x + 2t\,x\cdot y + t^2\, y\cdot y\}^2$$

$$\therefore g'(t) = 2\{x\cdot x + 2t\,x\cdot y + t^2\, y\cdot y\} \times \{2x\cdot y + 2t\, y\cdot y\}$$

$$\therefore g'(0) = 2\{x\cdot x\} \times \{2x\cdot y\}$$

$$\Rightarrow f'(x;y) = 4\,\|x\|^2\, x\cdot y$$

Answer.

Example ③ Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be given linear transformation. compute the derivative $f'(x;y)$ for the scalar field defined on $\mathbb{R}^n$ by the equation $f(x) = x \cdot T(x)$

Solution:→

$$\because f'(x;y) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+hy)\cdot T(x+hy) - x\cdot T(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+hy)\cdot \{T(x) + hT(y)\} - x\cdot T(x)}{h}$$

$$= \lim_{h\to 0} \frac{x\cdot T(x) + h\, y\cdot T(x) + h\, x\cdot T(y) + h^2 y\cdot T(y) - x\cdot T(x)}{h}$$

$$= \lim_{h\to 0} \frac{h\, y\cdot T(x) + h\, x\cdot T(y) + h^2\, y\cdot T(y)}{h}$$

$$= \lim_{h\to 0} \left\{ y\cdot T(x) + x\cdot T(y) + h\, y\cdot T(y) \right\}$$

$$f'(x;y) = y\cdot T(x) + x\cdot T(y)$$

Answer.

Directional derivatives and continuity :→

The next example describes a scalar field which has a directional derivative in every direction at O but which is not continuous at O.

Example :→ Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{if } x \neq 0, \quad f(0,y) = 0.$$

Let $a = (0,0)$ and let $y = (y_1, y_2)$ be any unit vector in $\mathbb{R}^2$.

If $y_1 \neq 0$ and $h \neq 0$, we have

$$f'(a;y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(hy) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(hy)}{h} = \lim_{h\to 0} \frac{f(hy_1, hy_2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3\, y_1 \cdot y_2^2}{h^2(y_1^2 + h^2 y_2^4)}}{h}$$

$$= \lim_{h\to 0} \frac{y_1 \cdot y_2^2}{y_1^2 + h^2 y_2^4} = \frac{y_1 \cdot y_2^2}{y_1^2} = \frac{y_2^2}{y_1}$$

$$\therefore f'(0;y) = \frac{y_2^2}{y_1}.$$

If $y = (0, y_2)$, $h \neq 0$, then

$$f'(0;y) = \lim_{h\to 0} \frac{f(0+hy) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(hy) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(0,hy_2)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$\therefore f'(0;y) = 0$

Therefor $f'(0;y)$ <u>exists</u> for all directions y.

Also, $f(x) \to 0$ as $x \to 0$ along any straight line through the origin.

If $x = (x_1, x_2) \to (0,0)$ along $x = y^2$ (except the origin) and $y \to 0$,

$$\therefore \lim_{\substack{x=y^2 \\ y\to 0}} f(x,y) = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

Since such points <u>exists</u> arbitrary close to the <u>origin</u> & since $f(0)=0$.

$\therefore f$ is <u>not</u> <u>continuous</u> at $(0,0)$.

<u>Remark</u>:→ The existance of all <u>directional</u> <u>derivatives</u> at a point $\not\Rightarrow$ <u>continuity</u> at that Point.

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The total derivative :→

If $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. Let $a \in D$.

If $f'(a)$ exists and let $E(a,h)$ denote the difference

$$E(a,h) = \frac{f(a+h) - f(a)}{h} - f'(a) \quad \text{if } h \neq 0 \quad -①$$

We define $E(a,0) = 0$. From ① we obtain the formula

$$f(a+h) = f(a) + h f'(a) + h E(a,h).$$

an equation which holds also for $h = 0$.

This is the first-order Taylor formula for ~~appo~~ approximating $f(a+h) - f(a)$ by $f'(a)h$. The ~~Error~~ Error committed is $hE(a,h)$.

From ① we see that

$$E(a,h) \longrightarrow 0 \quad \text{as } h \to 0.$$

Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field ~~fine~~ defined on a set S in $\mathbb{R}^n$. Let a be an interior point of S and let $B(a; r)$ be an n-ball lying in S. Let v be a vector with $\|v\| < r$, so that $a + v \in B(a; r)$

Functions of two variables

DEFINITION (Differentiable Scalar field) :→

Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. Then we say that f is differentiable at $a = (a_1, \cdots, a_n)$ if there exists a linear transformation

$$T_a : \mathbb{R}^n \longrightarrow \mathbb{R}$$

from $\mathbb{R}^n$ to $\mathbb{R}$, and a scalar function $E(a, v)$ such that

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v) \quad \text{— ②}$$

for $\|v\| < r$. Where $E(a,v) \to 0$ as $\|v\| \to 0$.

The linear transformation T_a is called the total derivative of f at a.

THEOREM :→ Assume f is differentiable at a with total derivative T_a. Then the derivative $f'(a;y)$ exists for every $y \in \mathbb{R}^n$ and we have

$$T_a(y) = f'(a;y) \quad \text{—— ①}$$

Moreover, $f'(a;y)$ is a scalar combination of the components of y. In fact, if $y = (y_1, \cdots, y_n)$ we have

$$f'(a;y) = \sum_{k=1}^{n} D_k f(a) \cdot y_k. \quad \text{—— ②}$$

Proof:→

Equation ① holds if $y=0$.

since $T_a(0)=0$ and $f'(a;0)=0$.

∴ We can assume that $y \neq 0$.

∵ f is differentiable at a we have a

Taylor formula:

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v) \quad \text{—— ③}$$

for $\|v\| < r$ for some $r>0$.

Where $E(a,v) \to 0$ as $\|v\| \to 0$.

Taking $v = hy$, where $h \neq 0$ & $|h|\,\|y\| < r$.

Then $\|v\| < r$.

∵ T_a is linear we have

$$T_a(v) = T_a(hv) = hT_a(v)$$

∴ Equation ③ becomes

$$\frac{f(a+hy) - f(a)}{h} = T_a(y) + \frac{|h|\,\|y\|}{h} E(a,v) \quad \text{— ④}$$

∵ $\|v\| \to 0$ as $h \to 0$ and since $\frac{|h|}{h} = \pm 1$

~~④ becomes~~

$$\therefore \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h} = T_a(y)$$

$$\Rightarrow f(a;y) = T_a(y)$$ proved.

$\because T_a$ is linear. And if $y = (y_1, \text{------}, y_n)$

We have $y = \sum_{k=1}^{n} y_k e_k$

$$\therefore T_a(y) = T_a\left(\sum_{k=1}^{n} y_k e_k\right)$$

$$= \sum_{k=1}^{n} y_k T_a(e_k)$$

$$= \sum_{k=1}^{n} y_k f'(a; e_k)$$

$$T_a(y) = \sum_{k=1}^{n} y_k D_k f(a)$$

proved.

The gradient of a scalar field :→

$$\because f'(a, y) = \sum_{k=1}^{n} D_k f(a) y_k = \nabla f(a) \cdot y.$$

Where $\nabla f(a)$ is the vector whose components are the partial derivatives of f at a,

$$\nabla f(a) = (D_1 f(a), \text{------}, D_n f(a))$$

is called gradient of f.

Remark :→
There for ③ becomes.

$$f(a + v) = f(a) + \nabla f(a) \cdot v + \|v\| E(a; v) \quad \text{---} \quad ④$$

Where $E(a, v) \longrightarrow 0$ as $\|v\| \longrightarrow 0$

THEOREM :→ If a scalar field f is differentiable at a, then f is continuous at a.

Proof:→

$\because f$ is differentiable at a then,

$$f(a+v) = f(a) + \nabla f(a)\cdot v + \|v\| E(a,v) \quad \leftarrow ①$$

for $\|v\| < r$ and $E(a,v) \to 0$ as $\|v\| \to 0$.

$$\therefore |f(a+v) - f(a)| = |\nabla f(a)\cdot v + \|v\| E(a,v)|$$

$$\leq |\nabla f(a)\cdot v| + |\|v\| E(a,v)|$$

(By tringle inequality)

$$\leq \|\nabla f(a)\| \|v\| + \|v\| |E(a,v)|$$

(By cauchy-schwarz inequality)

$$\therefore f(a+v) \longrightarrow f(a) \quad \text{as } \|v\| \to 0$$

$\therefore f$ is continuous at a.

proved.

Functions of two variables

Sufficient condition for differentiability :→

If $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in S$, then all partial derivatives $D_1 f(a), \ldots\ldots, D_n f(a)$ exist. But not conversely.

Example ① Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{if } x \neq 0, \; f(0,y) = 0$$

For this function, both partial derivatives $D_1 f(0)$ and $D_2 f(0)$ exists but f is not continuous at 0, hence f is not differentiable at 0.

THEOREM :→ (A sufficient condition for differentiability)

Assume that the partial derivatives $D_1 f, \ldots\ldots, D_n f$ exists in some n-ball $B(a)$ and are continuous at a. Then f is differentiable at a.

Proof :→

We have to show that

$$f(a+v) - f(a) = \nabla f(a) \cdot v + \|v\| \, E(a,v).$$

Where $E(a,v) \to 0$ as $\|v\| \to 0$.

(Where the only candidate for $T_a(v)$ is $\nabla f(a) \cdot v$)

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Let $\lambda = \|v\|$.

Then $v = \lambda u$, where $\|u\| = 1$.

We can choose λ small enough so that $a + v \in B(a)$ in which $D_1 f, \ldots, D_n f$ exists.

$\because u = u_1 e_1 + \cdots + u_n e_n$

Where $e_1, \ldots, e_n$ are the unit-coordinate vectors.

$$\therefore f(a+v) - f(a) = f(a+\lambda u) - f(a)$$

$$= \sum_{k=1}^{n} \{ f(a + \lambda v_k) - f(a + \lambda v_{k-1}) \}, \quad \text{—— (1)}$$

Where $v_0, v_1, \ldots, v_n$ are ~~vectors~~ any vectors in $\mathbb{R}^n$ s.t. $v_0 = 0$ and $v_n = u$.

We choose these vectors so they satisfy the recurrence relation

$v_k = v_{k-1} + u_k e_k$. That is, we take

$v_0 = 0, \quad v_1 = u_1 e_1, \quad v_2 = u_1 e_1 + u_2 e_2, \ldots$

$\ldots, v_n = u_1 e_1 + \cdots + u_n e_n$.

$\therefore$ From equation (1),

$$f(a+v) - f(a) = \sum_{k=1}^{n} \{ f(a + \lambda v_{k-1} + \lambda u_k e_k) - f(a + \lambda v_{k-1}) \}$$

$$= \sum_{k=1}^{n} \{ f(b_k + \lambda u_k e_k) - f(b_k) \} \quad \text{—— (2)}$$

Where $b_k = a + \lambda v_{k-1}$.

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$\because$ b_k and $b_k + \lambda u_k e_k$ differ only in their kth component.

$\therefore$ By mean value theorem, we have

$$f(b_k + \lambda u_k e_k) - f(b_k) = \lambda u_k D_k f(c_k)$$

where c_k lies on the line segment joining b_k and $b_k + \lambda u_k e_k$.

When $\lambda \to 0$ then $b_k \to a \Rightarrow c_k \to a$.

putting these value in ②

$$f(a+v) - f(a) = \sum_{k=1}^{n} \lambda u_k D_k f(c_k)$$

$$= \lambda \sum_{k=1}^{n} D_k f(c_k) \cdot u_k$$

$\because$ $\nabla f(a) \cdot v = \lambda \nabla f(a) \cdot u = \lambda \sum_{k=1}^{n} D_k f(c_k) u_k$. so

$$f(a+v) - f(a) - \nabla f(a) \cdot v$$

$$= \lambda \sum_{k=1}^{n} \{D_k f(c_k) - D_k f(a)\} u_k$$

$$= \|v\| E(a, v)$$

Where $E(a,v) = \sum_{k=1}^{n} \{D_k f(c_k) - D_k f(a)\} u_k$

$\because$ $c_k \to a$ as $\|v\| \to 0$.

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$\because$ Each partial derivatives $D_k f$ is continuous at a, we get

$E(a, v) \longrightarrow 0$ as $\|v\| \to 0$.

$\therefore f$ is differentiable at a.

proved.

Ex(1) Find the gradient vector at each point at which its exists for the scalar field $f(x, y, z) = x^2 - y^2 + 2z^2$

Solution :→

$$\because \text{grad} f = \nabla f$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f(x, y, z)$$

$$= 2x\,\hat{i} - 2y\,\hat{j} + 4z\,\hat{k}$$

Answer.

Derivative ~~in~~ of vector field

Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector field defined on a subset S of $\mathbb{R}^n$.

If a is an interior point point of S and if y be any vector of $\mathbb{R}^n$ then we define the derivative $f'(a;y)$ by

$$f'(a;y) = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h} \quad —①$$

Whenever the limit exists.

The derivative $f'(a;y)$ is a vector in $\mathbb{R}^m$.

$$\text{i,e; } f'(a;y) = \left(f_1'(a;y), \text{------}, f_m'(a;y)\right)$$

$$= \sum_{k=1}^{m} f_k'(a;y)\, e_k \quad —②$$

DEFINITION :→ Let $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector field. Then f is said to be differentiable at an interior point a if $\exists$ a linear transformation

$$T_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ s.t.}$$

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v)$$

Where $E(a,v) \longrightarrow 0$ as $v \to 0$.

The linear transformation ~~is called~~ T_a is called total derivative of f at a.

THEOREM :→ Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$. If f is differentiable at a with total derivative T_a. Then the derivative $f'(a;y)$ exists for every y in $\mathbb{R}^n$, and we have

$$T_a(y) = f'(a;y) \quad \text{—— ①}$$

Moreover; if $f = (f_1, \cdots, f_m)$ and if $y = (y_1, \cdots, y_n)$, we have

$$T_a(y) = \sum_{k=1}^{m} \nabla f_k(a) \cdot y \, e_k$$

$$= \left(\nabla f_1(a) \cdot y, \cdots\cdots, \nabla f_m(a) \cdot y\right) \quad \text{— ②}$$

Proof:→ We argu in the scalar case.

If $y = 0$, then $f'(a;y) = 0$ and $T_a(0) = 0$.

∴ We assume that $y \neq 0$.

Taking $v = hy$

$$\therefore f(a+hy) - f(a) = T_a(hy) + \|hy\| E(a,v)$$

$$= h\,T_a(y) + |h|\,\|y\|\,E(a,v)$$

$$\therefore \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h} = T_a(y)$$

$$\Rightarrow f'(a;y) = T_a(y)$$

proved ①

To prove ② we simply note that

$$f'(a;y) = \sum_{k=1}^{n} f_k'(a;y)\, e_k$$

$$= \sum_{k=1}^{m} \nabla f_k(a)\cdot y\, e_k$$

proved.

Remark :→

Equation ② can also written as

$$T_a(y) = Df(a)y,$$

Where $Df(a)$ is $m\times n$ matrix whose kth row is $\nabla f_k(a)$, and y ~~regard as~~ is regarded as an $n\times 1$ ~~matrix~~ column matrix.

The matrix is called the Jacobian matrix of f at a.

$$Df(a) = \begin{bmatrix} D_1 f_1(a) & D_2 f_1(a) & \cdots\cdots & D_n f_1(a) \\ D_1 f_2(a) & D_2 f_2(a) & \cdots\cdots & D_n f_2(a) \\ \vdots & & & \\ D_1 f_m(a) & D_2 f_m(a) & \cdots\cdots & D_n f_m(a) \end{bmatrix}_{m\times n}.$$

The total derivative T_a is also written as $f'(a)$. The derivative $f'(a)$ is a linear transfor-mation; The Jacobian $Df(a)$ is a matrix representation for this transformation.

THANK YOU